Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-04-17

Conceptual Review

Question 1. In the below, f, g denote scalar fields while $\mathbf{F}, \mathbf{G}$ denote vector fields. Which of these expressions do not make sense? For the ones that do make sense, do they output a scalar field or a vector field?

(a) $\operatorname{grad} f$ — This is fine, outputs a vector field.

Rmk: ∇ is the "vector" $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ so "$\nabla \cdot \langle P, Q, R\rangle$" means $P_x + Q_y + R_z$, i.e. divergence.

~~(b)~~ $\operatorname{curl}(\underbrace{\operatorname{div} \mathbf{F}}_{\text{scalar field}})$ — but curl takes a vector field argument.

(c) $\operatorname{grad}(\underbrace{\operatorname{div}(\operatorname{curl} \mathbf{F})}_{\text{always 0 scalar field}}) = \vec{0}$ zero vector field

~~(d)~~ $\overset{\text{grad}}{\nabla}(\nabla \overset{\text{curl}}{\times} f)$ — curl takes in a vector (and even if that part made sense, $\nabla(\ldots)$ only works on scalar fields)

(e) $\underbrace{\overset{\text{div}}{\nabla \cdot} (\nabla \times}_{\text{outputs 0 scalar field.}} (f\mathbf{G}))$ outputs a scalar field $= 0$

(f) $\nabla \cdot (\nabla f \times \nabla g) = \nabla g \cdot (\underbrace{\nabla \times \nabla f}_{\text{outputs } \vec{0}}) - \nabla f \cdot (\underbrace{\nabla \times \nabla g}_{\text{ditto}}) = 0$ scalar field.

Question 2. Of the expressions that do make sense above, are there any that are *always* equal to zero (either the zero scalar field or the zero vector field)?

Hint: The vector identity $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$ may be useful for one of the expressions.

Problems

Problem 1. Let n be a constant, let r denote the scalar field $\sqrt{x^2+y^2+z^2}$, and let $\mathbf{r}$ denote the vector field $\langle x, y, z\rangle$.

Compute the following:

(a) $\nabla(r^n)$
(b) $\nabla \times (r^{n-1}\mathbf{r})$
(c) $\nabla \cdot (r^{n-1}\mathbf{r})$.

For part (c), what value of n makes the answer equal to zero?

$$\nabla \times \langle P,Q,R\rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{bmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

(a) Note r^2 a little more convenient than r

$$\nabla(r^n) = \nabla\left((r^2)^{n/2}\right) = \frac{n}{2}(r^2)^{\frac{n}{2}-1}\nabla(r^2) = \boxed{n r^{n-2}\vec{r}}.$$

$$\nabla\left(\frac{r^{n+1}}{n+1}\right) = r^{n-1}\vec{r}$$

(b) $$\nabla \times (r^{n-1}\vec{r}) = \underbrace{\nabla(r^{n-1}) \times \vec{r}}_{=\vec{0}} + r^{n-1}\underbrace{(\nabla \times \vec{r})}_{=\vec{0} \text{ by direct comp}} = \boxed{\vec{0}}$$

(c) $$\nabla \cdot (r^{n-1}\vec{r}) = \nabla(r^{n-1}) \cdot \vec{r} + r^{n-1}\,\nabla \cdot \vec{r} = (n-1)r^{n-3}\underbrace{\vec{r}\cdot\vec{r}}_{r^2} + r^{n-1}\underbrace{\nabla\cdot\vec{r}}_{3}$$

$$= (n-1)r^{n-1} + r^{n-1}3$$

$$= \boxed{(n+2)r^{n-1}}$$

So $\nabla \cdot (r^{n-1}\vec{r}) = 0$ (zero scalar field) when $n = -2$. (This is the exponent encountered when dealing w/ electromagnetism, gravity, etc...)